SUT Journal of Mathematics
Vol. 43, No. 2 (2007), 137–147

On the simultaneous confidence procedure for multiple comparisons with a control

Takahiro Nishiyama

(Received October 2, 2007)

Abstract. In this paper, we consider the simultaneous confidence procedure for multiple comparisons with a control among mean vectors from the multivariate normal distributions. Seo[9] proposed a conservative simultaneous confidence procedure for multiple comparisons with a control. Further, Seo[9] conjectured that this procedure always yields the conservative simultaneous confidence intervals. In this paper, we give the affirmative proof of this conjecture in the case of four mean vectors. We also give the upper bound for the conservativeness of the procedure. Finally, numerical results by Monte Carlo simulation are given.

AMS 2000 *Mathematics Subject Classification.* 62H10, 62E17.

Key words and phrases. Comparisons with a control, conservativeness, coverage probability, Monte Carlo simulation.

§1. Introduction

Simultaneous confidence procedures for multiple comparisons among mean vectors have been studied by many authors. In many experimental situations, pairwise comparisons and comparisons with a control are standard for multiple comparisons. On the univariate case, a number of multiple comparison procedure for pairwise comparisons and comparisons with a control have been proposed for balanced and unbalanced cases (see, e.g., Hochberg and Tamhane[5]). In one of these procedures, Tukey-Kramer (TK) procedure, which was proposed by Tukey[14] and Kramer[6][7], is well known as a typical procedure. In one of the important properties of TK procedure, this procedure yields the conservative simultaneous confidence intervals for all pairwise comparisons among means (see, e.g., Benjamini and Braun[1]). This property is known as the generalized Tukey conjecture. For the theoretical discussions to prove the generalized Tukey conjecture, see Hayter[3][4], Brown[2] and so on. Seo, Mano and Fujikoshi[11] proposed the multivariate Tukey-Krmaer (MTK) procedure. For the MTK procedure, the multivariate generalized Tukey conjecture has been affirmatively proved in the case of three correlated mean vectors. Recently, Nishiyama and Seo[8] gave the affirmative proof of the conjecture in the case of four mean vectors. Further, relating to the conjecture, Seo[10] and Seo and Nishiyama[12] discussed the upper bound for the conservativeness of the MTK procedure.

In the case of comparisons with a control, concerning to the MTK procedure, Seo[9] proposed a conservative simultaneous confidence procedure. In the case of three correlated mean vectors, its conservativeness has been affirmatively proved by Seo[9], and Seo and Nishiyama[12] gave the upper bound for the conservativeness of this procedure.

In this paper, we discuss the conservativeness of the simultaneous confidence procedure for comparisons with a control in the case of four correlated mean vectors. Further, we give the upper bound for the conservativeness of the procedure. The organization of the paper is as follows; in Section 2, we describe the conservative simultaneous confidence procedure for comparisons with a control. Also, the conservativeness of the procedure in the case of four mean vectors and its upper bound for the conservativeness are given. In Section 3, some numerical results by Monte Carlo simulation are given.

§2. Conservative simultaneous confidence procedure for multiple comparisons with a control

Let $\boldsymbol{M} = [\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k]$ be the unknown $p \times k$ matrix of k mean vectors corresponding to the k treatments, where $\boldsymbol{\mu}_i$ is the mean vector from ith population. Here, we assume that k-th treatment is a control treatment. And let $\widehat{\boldsymbol{M}} = [\widehat{\boldsymbol{\mu}}_1, \dots, \widehat{\boldsymbol{\mu}}_k]$ be an estimator of $\boldsymbol{M}$ such that $\mathrm{vec}(\boldsymbol{X})$ has $\mathrm{N}_{kp}(\boldsymbol{0}, \boldsymbol{V} \otimes \boldsymbol{\Sigma})$, where $\boldsymbol{X} = \widehat{\boldsymbol{M}} - \boldsymbol{M}$, $\boldsymbol{V} = [v_{ij}]$ is a known $k \times k$ positive definite matrix and $\boldsymbol{\Sigma}$ is an unknown $p \times p$ positive definite matrix, and $\mathrm{vec}(\cdot)$ denotes the column vector formed by stacking the columns of the matrix under each other. Further, we assume that $\boldsymbol{S}$ is an unbiased estimator of $\boldsymbol{\Sigma}$ such that $\nu \boldsymbol{S}$ is independent of $\widehat{\boldsymbol{M}}$ and is distributed as a Wishart distribution $\mathrm{W}_p(\boldsymbol{\Sigma}, \nu)$. Then we have the simultaneous confidence intervals for comparisons with a control among mean vectors given by

$$\text{(2.1)} \quad \boldsymbol{a}'\boldsymbol{M}\boldsymbol{b} \in \Big[\, \boldsymbol{a}'\widehat{\boldsymbol{M}}\boldsymbol{b} \pm t(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b})^{1/2}(\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2} \Big], \quad \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ \forall \boldsymbol{b} \in \mathbb{B},$$

where $\mathbb{R}^p - \{\boldsymbol{0}\}$ is a set of any nonzero real p-dimensional vectors, $\mathbb{B}$ is a subset in the k-dimensional space such that

$$\mathbb{B} = \{\boldsymbol{b} \in \mathbb{R}^k : \boldsymbol{b} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \dots, k-1\},$$

$\boldsymbol{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a k-dimensional unit vector which having 1 at i-th component, t is the upper α percentile of $T^2_{\max\cdot\mathrm{c}}$ statistic defined by

$$\begin{aligned} T^2_{\max\cdot\mathrm{c}} &= \max_{\boldsymbol{b}\in\mathbb{B}} \left\{ \frac{(\boldsymbol{X}\boldsymbol{b})'\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}} \right\} \\ &= \max_{i=1,\dots,k-1} \left\{ (\boldsymbol{x}_i - \boldsymbol{x}_k)'(d_{ik}\boldsymbol{S})^{-1}(\boldsymbol{x}_i - \boldsymbol{x}_k) \right\}, \end{aligned}$$

and $d_{ik} = v_{ii} - 2v_{ik} + v_{kk}$.

Also, we can express (2.1) as

$$\boldsymbol{a}'(\boldsymbol{\mu}_i - \boldsymbol{\mu}_k) \in \left[\ \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}_i - \widehat{\boldsymbol{\mu}}_k) \pm t\ (d_{ik}\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2} \right], \quad \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ 1 \le i \le k-1.$$

Then for $k \ge 3$, Seo[9] proposed a conservative procedure by replacing with $t_{\mathrm{c}\cdot V_1}$ as an approximation to t, and conjectured conservative simultaneous confidence intervals given by

$$\boldsymbol{a}'(\boldsymbol{\mu}_i - \boldsymbol{\mu}_k) \in \left[\boldsymbol{a}'\,(\widehat{\boldsymbol{\mu}}_i - \widehat{\boldsymbol{\mu}}_k) \pm t_{\mathrm{c}\cdot V_1} \sqrt{d_{ik}\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}}\right], \quad \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ 1 \le i \le k-1, \tag{2.2}$$

where $t^2_{\mathrm{c}\cdot V_1}$ is the upper α percentile of $T^2_{\max\cdot\mathrm{c}}$ statistic with $\boldsymbol{V} = \boldsymbol{V}_1$ and $\boldsymbol{V}_1$ satisfies with the conditions $d_{ij} = d_{ik} + d_{jk},\ 1 \le i < j \le k-1$. We note that the matrix $\boldsymbol{V}_1$ satisfies with $d_{12} = d_{13} + d_{23}$ for the case $k = 3$. By a reduction of relating to the coverage probability of (2.2), Seo[9] proved that the coverage probability in the case $k = 3$ is equal or greater than $1 - \alpha$ for any positive definite matrix $\boldsymbol{V}$. Besides, Seo and Nishiyama[12] discussed the bound of conservative simultaneous confidence level. Unfortunately, this conjecture is not proved in the case $k \ge 4$, so we attempt to prove this conjecture and give the upper bound for the conservativeness in the case $k = 4$. We note that the matrix $\boldsymbol{V}_1$ satisfies with $d_{12} = d_{14} + d_{24}$, $d_{13} = d_{14} + d_{34}$ and $d_{23} = d_{24} + d_{34}$ for the case $k = 4$.

First of all, we consider the probability

$$Q(q, \boldsymbol{V}, \mathbb{B}) = \Pr\left\{ (\boldsymbol{X}\boldsymbol{b})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{X}\boldsymbol{b}) \le q(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}),\ \forall \boldsymbol{b} \in \mathbb{B} \right\}, \tag{2.3}$$

where q is any fixed constant. Without loss of generality, we assume $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. When $q = t^*_{\mathrm{c}} (\equiv t^2_{\mathrm{c}\cdot V_1}/\nu)$, the coverage probability (2.3) is the same as one of (2.2). The conservativeness of the simultaneous confidence intervals (2.2) means that $Q(t^*_{\mathrm{c}}, \boldsymbol{V}, \mathbb{B}) \ge Q(t^*_{\mathrm{c}}, \boldsymbol{V}_1, \mathbb{B}) = 1 - \alpha$, then the following theorem for the case $k = 3$ is given by Seo and Nishiyama[12].

Theorem 1. *(Seo and Nishiyama[12]) Let $Q(q, \boldsymbol{V}, \mathbb{B})$ be the coverage probability for (2.3) with a known matrix $\boldsymbol{V}$ for the case $k = 3$. Then*

$$1 - \alpha = Q(t_{\mathrm{c}}^*, \boldsymbol{V}_1, \mathbb{B}) \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{B}) < Q(t_{\mathrm{c}}^*, \boldsymbol{V}_2, \mathbb{B})$$

holds for any positive definite matrix $\boldsymbol{V}$, where $t_{\mathrm{c}}^* = t_{\mathrm{c}\cdot V_1}^2/\nu$, $\mathbb{B} = \{\boldsymbol{b} \in \mathbb{R}^k : \boldsymbol{b} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}$ and $\boldsymbol{V}_1$ satisfies with $d_{12} = d_{13} + d_{23}$ and $\boldsymbol{V}_2$ satisfies with $\sqrt{d_{12}} = |\sqrt{d_{13}} - \sqrt{d_{23}}|$.

In connection with Theorem 1, we prepare the following conjecture for the case $k \geq 4$.

Conjecture 1. *Let $Q(q, \boldsymbol{V}, \mathbb{B})$ be the coverage probability for (2.3) with a known matrix $\boldsymbol{V}$. Then*

$$1 - \alpha = Q(t_{\mathrm{c}}^*, \boldsymbol{V}_1, \mathbb{B}) \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{B}) < Q(t_{\mathrm{c}}^*, \boldsymbol{V}_2, \mathbb{B})$$

holds for any positive definite matrix $\boldsymbol{V}$, where $t_{\mathrm{c}}^* = t_{\mathrm{c}\cdot V_1}^2/\nu$, $\mathbb{B} = \{\boldsymbol{b} \in \mathbb{R}^k : \boldsymbol{b} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}$ and $\boldsymbol{V}_1$ satisfies with $d_{ij} = d_{ik} + d_{jk}$ for all $i, j (1 \leq j \leq k-1)$ and $\boldsymbol{V}_2$ satisfies with $\sqrt{d_{ij}} = |\sqrt{d_{ik}} - \sqrt{d_{jk}}|$ for all $i, j (1 \leq j \leq k-1)$.

Now, we discuss the case of $k = 4$ in Conjecture 1. We obtain the following result by an extension of the idea in Seo[9] and Seo and Nishiyama[12].

Theorem 2. *Let $Q(q, \boldsymbol{V}, \mathbb{B})$ be the coverage probability for (2.3) with a known matrix $\boldsymbol{V}$ for the case $k = 4$. Then*

$$1 - \alpha = Q(t_{\mathrm{c}}^*, \boldsymbol{V}_1, \mathbb{B}) \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{B}) < Q(t_{\mathrm{c}}^*, \boldsymbol{V}_2, \mathbb{B})$$

holds for any positive definite matrix $\boldsymbol{V}$, where $t_{\mathrm{c}}^* = t_{\mathrm{c}\cdot V_1}^2/\nu$, $\mathbb{B} = \{\boldsymbol{b} \in \mathbb{R}^k : \boldsymbol{b} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}$ and $\boldsymbol{V}_1$ satisfies with $d_{12} = d_{14} + d_{24}$, $d_{13} = d_{14} + d_{34}$ and $d_{23} = d_{24} + d_{34}$, and $\boldsymbol{V}_2$ satisfies with $\sqrt{d_{12}} = |\sqrt{d_{14}} - \sqrt{d_{24}}|$, $\sqrt{d_{13}} = |\sqrt{d_{14}} - \sqrt{d_{34}}|$ and $\sqrt{d_{23}} = |\sqrt{d_{24}} - \sqrt{d_{34}}|$.

Proof. Let $\boldsymbol{A}$ be $k \times k$ nonsingular matrix such that $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$. Then by the transformation from $\boldsymbol{X}$ to $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{A}^{-1}$, we have $\mathrm{vec}(\boldsymbol{Y}) \sim \mathrm{N}_{kp}(\boldsymbol{0}, \boldsymbol{I}_k \otimes \boldsymbol{I}_p)$. Let

$$\Gamma = \left\{\gamma \in \mathbb{R}^k;\ \gamma = (\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b})^{-1/2}\boldsymbol{A}\boldsymbol{b},\ \boldsymbol{b} \in \mathbb{B}\right\},$$

which is a subset of unit vector in $\mathbb{R}^k$. Then we can rewrite the coverage probability $Q(q, \boldsymbol{V}, \mathbb{B})$ as

$$\begin{aligned} Q(q, \boldsymbol{V}, \mathbb{B}) &= \Pr\left\{(\boldsymbol{Y}\boldsymbol{A}\boldsymbol{b})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{Y}\boldsymbol{A}\boldsymbol{b}) \leq q(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}),\ \forall \boldsymbol{b} \in \mathbb{B}\right\} \\ &= \Pr\left\{(\boldsymbol{Y}\boldsymbol{\gamma})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{Y}\boldsymbol{\gamma}) \leq q,\ \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\right\}. \end{aligned}$$

Further, we consider the transformation from $\boldsymbol{S}$ to $\boldsymbol{L} = \mathrm{diag}(\ell_1, \ldots, \ell_p)$, $\ell_1 \geq \cdots \geq \ell_p$ and a $p \times p$ orthogonal matrix $\boldsymbol{H}_1$ such that $\nu \boldsymbol{S} = \boldsymbol{H}_1 \boldsymbol{L} \boldsymbol{H}_1'$. It is well known (see, e.g., Siotani, Hayakawa and Fujikoshi[13]) that $\boldsymbol{L}$ and $\boldsymbol{H}_1$ are independent. Then

$$Q(q, \boldsymbol{V}, \mathbb{B}) = \mathrm{E}_L \left[\Pr \left\{ (\boldsymbol{Y}\boldsymbol{\gamma})' \boldsymbol{L}^{-1} (\boldsymbol{Y}\boldsymbol{\gamma}) \leq q,\ \boldsymbol{\gamma} \in \boldsymbol{\Gamma} \right\} \right].$$

Since the dimension of the space spanned by $\mathbb{B}$ equals 3, there exists a $k \times k$ orthogonal matrix $\boldsymbol{H}_2$ such that

$$\boldsymbol{\gamma}_m' \boldsymbol{H}_2 = \left[\boldsymbol{\delta}_m',\ 0 \right],\ m = 1, 2, 3,$$

where $\boldsymbol{\delta}_m = (\delta_{m1}, \delta_{m2}, \delta_{m3})'$ is a 3-dimensional vector. Here $\boldsymbol{\delta}_m$ satisfies $\boldsymbol{\delta}_m' \boldsymbol{\delta}_m = 1$, so we can write

$$\boldsymbol{\delta}_m = \begin{pmatrix} \sin\beta_{m1} \sin\beta_{m2} \\ \sin\beta_{m1} \cos\beta_{m2} \\ \cos\beta_{m1} \end{pmatrix},\ m = 1, 2, 3,$$

where $0 \leq \beta_{m1} < \pi$ and $0 \leq \beta_{m2} < 2\pi$.

Further, we can write $\boldsymbol{Y}\boldsymbol{H}_2 = [\boldsymbol{U}, \widetilde{\boldsymbol{U}}]$, where $\boldsymbol{U}$ is a $p \times 3$ matrix. Letting $\boldsymbol{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p]'$, where

$$\boldsymbol{u}_s = ||\boldsymbol{u}_s|| \begin{pmatrix} \sin\theta_{s1} \sin\theta_{s2} \\ \sin\theta_{s1} \cos\theta_{s2} \\ \cos\theta_{s1} \end{pmatrix} = r_s \begin{pmatrix} \sin\theta_{s1} \sin\theta_{s2} \\ \sin\theta_{s1} \cos\theta_{s2} \\ \cos\theta_{s1} \end{pmatrix},\ s = 1, \ldots, p,$$

and r_s^2, θ_{s1} and θ_{s2} are independently distributed as χ^2 distribution with three degrees of freedom, uniform distribution on $\mathrm{U}[0, \pi)$ and on $\mathrm{U}[0, 2\pi)$, respectively. Then the coverage probability can be written as

$$\begin{aligned} Q(q, \boldsymbol{V}, \mathbb{B}) = \mathrm{E}_{L,R} \Bigg[\Pr \Bigg\{ \sum_{s=1}^{p} \frac{r_s^2}{\ell_s} (&\sin\theta_{s1} \sin\theta_{s2} \sin\beta_{m1} \sin\beta_{m2} \\ &+ \sin\theta_{s1} \cos\theta_{s2} \sin\beta_{m1} \cos\beta_{m2} + \cos\theta_{s1} \cos\beta_{m1})^2 \leq q \text{ for } m = 1, 2, 3 \Bigg\} \Bigg], \end{aligned}$$

where $\boldsymbol{R} = \mathrm{diag}(r_1, \ldots, r_p)$ is independent of $\boldsymbol{L} = \mathrm{diag}(\ell_1, \ldots, \ell_p)$.

Relating the coverage probability $Q(q, \boldsymbol{V}, \mathbb{B})$, we consider the probability

$$\begin{aligned} (2.4) \quad G(\boldsymbol{\beta}) = \Pr \Bigg[\sum_{s=1}^{p} \frac{r_s^2}{l_s} (&\sin\theta_{s1} \sin\theta_{s2} \sin\beta_{m1} \sin\beta_{m2} \\ &+ \sin\theta_{s1} \cos\theta_{s2} \sin\beta_{m1} \cos\beta_{m2} + \cos\theta_{s1} \cos\beta_{m1})^2 \leq q \text{ for } m = 1, 2, 3 \Bigg], \end{aligned}$$

where $\boldsymbol{\beta} = (\beta_{11}, \beta_{21}, \beta_{31}, \beta_{12}, \beta_{22}, \beta_{32})'$. Also, we define the volume Ω and D_m as follows.

$$\begin{aligned}\Omega = \ & \{(\theta_{s1}, \theta_{s2})^p : 0 < \theta_{s1} < \pi, 0 < \theta_{s2} < 2\pi, 1 \le s \le p\},\\ D_m = \ & \Bigg\{ (\theta_{s1}, \theta_{s2})^p \in \Omega : \sum_{s=1}^{p} \frac{r_s^2}{\ell_s} (\sin\theta_{s1} \sin\theta_{s2} \sin\beta_{m1} \sin\beta_{m2} \\ & \qquad + \sin\theta_{s1} \cos\theta_{s2} \sin\beta_{m1} \cos\beta_{m2} + \cos\theta_{s1} \cos\beta_{m1})^2 > q \Bigg\}.\end{aligned}$$

Then we note that the probability (2.4) is equal to 1–volume$[\cup_{m=1}^{3} D_m]/(2\pi^2)^p$. Therefore, to minimize $G(\boldsymbol{\beta})$ is equivalent to maximizing the value for volume of the union of D_m's. Similarly, to maximize $G(\boldsymbol{\beta})$ is equivalent to minimizing the value for volume of the union of D_m's.

Here, for comparisons with a control, we can assume that subset $\boldsymbol{b}$'s of the set $\mathbb{B}$ are as follows.

$$\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \ \boldsymbol{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \ \boldsymbol{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

At first, we consider the case that volume$[\cup_{m=1}^{3} D_m]$ is maximum. Assuming that $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are orthogonal, we can put

$$\boldsymbol{\delta}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{\delta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{\delta}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we can get $\beta_{11} = 0, \beta_{21} = \pi/2, \beta_{31} = \pi/2, \beta_{12} = 0, \beta_{22} = 0, \beta_{32} = \pi/2$.

For example, putting $p = 1$, $r_1^2/\ell_1 = 1$ and $q = 0.5$, we have

$$\begin{aligned}G(\boldsymbol{\beta}) = \Pr \Big[& (\sin\theta_{11} \sin\theta_{12} \sin\beta_{m1} \sin\beta_{m2} \\ & + \sin\theta_{11} \cos\theta_{12} \sin\beta_{m1} \cos\beta_{m2} + \cos\theta_{11} \cos\beta_{m1})^2 \le 0.5 \text{ for } m = 1, 2, 3 \Big],\end{aligned}$$

and

$$\begin{aligned}D_i \ = \ & \{(\theta_{11}, \theta_{12}) \in \Omega : [\sin\theta_{11} \sin\theta_{12} \sin\beta_{i1} \sin\beta_{i2} \\ + \ & \sin\theta_{11} \cos\theta_{12} \sin\beta_{i1} \cos\beta_{i2} + \cos\theta_{11} \cos\beta_{i1}]^2 > 0.5 \text{ for } i = 1, 2, 3\}.\end{aligned}$$

It is noted from Figure 1 that D_i's don't overlap, so the volume$[\cup_{i=1}^{3} D_i]$ is maximum when $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3$ are orthogonal.

On the other hands, in the case $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are not orthogonal, we choose

$$\boldsymbol{\delta}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{\delta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{\delta}_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Then we can get $\beta_{11} = 0, \beta_{21} = \pi/2, \beta_{31} = \pi/2, \beta_{12} = 0, \beta_{22} = 0, \beta_{32} = \pi/4$.

In this case, it is noted from Figure 2 that D_2 and D_3 overlap each other. So the volume$[\cup_{i=1}^3 D_i]$ is not maximum when $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are not orthogonal.

Hence, $Q(q, \boldsymbol{V}, \mathbb{B})$ is minimum when $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are orthogonal each other. Therefore, $\boldsymbol{\delta}_\ell' \boldsymbol{\delta}_m = 0 (\ell \neq m)$, that is, $\boldsymbol{\gamma}_\ell' \boldsymbol{\gamma}_m = 0 (\ell \neq m)$. We can show that $\boldsymbol{\gamma}_1' \boldsymbol{\gamma}_2 = 0$ if and only if $v_{12} - v_{24} - v_{14} + v_{44} = 0$. Therefore, we can get the condition $d_{12} = d_{14} + d_{24}$. For the case that $\boldsymbol{\gamma}_1' \boldsymbol{\gamma}_3 = 0$ and $\boldsymbol{\gamma}_2' \boldsymbol{\gamma}_3 = 0$, we can get the similar conditions $d_{13} = d_{14} + d_{34}$ and $d_{23} = d_{24} + d_{34}$. Thus, we can get the condition of $\boldsymbol{V}_1$ as $d_{ij} = d_{i4} + d_{j4} (1 \leq i < j \leq 3)$.

Secondly, we consider the case that volume$[\cup_{m=1}^3 D_m]$ is minimum. By using same procedure, we note that $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, and $\boldsymbol{\delta}_3$ are same in this case. So, $\boldsymbol{\delta}_\ell' \boldsymbol{\delta}_m = \boldsymbol{\delta}_\ell' \boldsymbol{\delta}_\ell = 1 (\ell \neq m)$, that is, $\boldsymbol{\gamma}_\ell' \boldsymbol{\gamma}_\ell = 1$. We can show that $\boldsymbol{\gamma}_1' \boldsymbol{\gamma}_2 = 1$ if and only if $v_{12} - v_{24} - v_{14} + v_{44} = \sqrt{d_{14}}\sqrt{d_{24}}$. Therefore, we can get the condition $\sqrt{d_{12}} = |\sqrt{d_{14}} - \sqrt{d_{24}}|$. For the case that $\boldsymbol{\gamma}_1' \boldsymbol{\gamma}_3 = 1$ and $\boldsymbol{\gamma}_2' \boldsymbol{\gamma}_3 = 1$, we can get the similar conditions $\sqrt{d_{13}} = |\sqrt{d_{14}} - \sqrt{d_{34}}|$ and $\sqrt{d_{23}} = |\sqrt{d_{24}} - \sqrt{d_{34}}|$. Thus, we can get the condition of $\boldsymbol{V}_2$ as $\sqrt{d_{ij}} = |\sqrt{d_{i4}} - \sqrt{d_{j4}}| (1 \leq i < j \leq 3)$.

We note that there does not exist $\boldsymbol{V}_2$ as a positive definite matrix. However, we can find $\boldsymbol{V}_2$ as a positive semi-definite matrix. Therefore, when $q = t_{\mathrm{c}}^* (\equiv t_{\mathrm{c}\cdot V_1}^2/\nu)$, we note that $1 - \alpha = Q(t_{\mathrm{c}}^*, \boldsymbol{V}_1, \mathbb{B}) \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{B}) < Q(t_{\mathrm{c}}^*, \boldsymbol{V}_2, \mathbb{B})$. □

§3. Numerical Examinations

This section gives some numerical results of the coverage probability for $T_{\max\cdot\mathrm{c}}^2$ statistic and the upper percentiles of the statistic by Monte Carlo simulation. The Monte Calro simulations are made from 10^6 trials for each of parameters based on normal random vectors from $\mathrm{N}_{kp}(\boldsymbol{0}, \boldsymbol{V} \otimes \boldsymbol{I}_p)$. Also, we note that the sample covariance matrix $\boldsymbol{S}$ is formed independently in each time with ν degrees of freedom.

Table 1 gives the simulation results for the case where $\alpha = 0.1, 0.5, 0.01; p = 1, 2, 5; k = 4; \nu = 20, 40, 60$; and $\boldsymbol{V} = \boldsymbol{I}, \boldsymbol{V}_1, \boldsymbol{V}_2$, that is,

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{V}_1 = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}, \boldsymbol{V}_2 = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 4 \end{bmatrix}.$$

Here we note that $\boldsymbol{V}_1$ is a positive definite matrix that satisfies $d_{ij} = d_{i4} + d_{j4} (1 \leq i < j \leq 3)$ and $\boldsymbol{V}_2$ is a positive semi-definite matrix that satisfies $\sqrt{d_{ij}} = |\sqrt{d_{i4}} - \sqrt{d_{j4}}| (1 \leq i < j \leq 3)$.

It can be seen from some simulation results in Table 1 that the upper percentiles with $\boldsymbol{V} = \boldsymbol{V}_1$ are always maximum values and those with $\boldsymbol{V} = \boldsymbol{V}_2$ are always minimum values for each parameters. Besides, the upper percentiles with $\boldsymbol{V} = \boldsymbol{I}$ are always between those with $\boldsymbol{V} = \boldsymbol{V}_1$ and those with $\boldsymbol{V} = \boldsymbol{V}_2$.

It is noted from Table 1 that we can obtain the upper bounds for the conservativeness of multiple comparisons with a control. For example, when $p = 2, \nu = 20$ and $\alpha = 0.1$, we note that $0.90 \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{B}) < 0.965$ for any positive definite $\boldsymbol{V}$. Further, it may be noted that the coverage probabilities do not depend on p and ν.

In conclusion, the conservative approximate procedure which is proposed by this paper is useful for the simultaneous confidence intervals estimation in the case of comparisons with a control.

p	ν	α	$\boldsymbol{V} = \boldsymbol{V}_1$	$\boldsymbol{V} = \boldsymbol{I}$	$\boldsymbol{V} = \boldsymbol{V}_2$	$\mathrm{Q}(t_c^*, \boldsymbol{I}, \mathbb{B})$	$\mathrm{Q}(t_c^*, \boldsymbol{V}_2, \mathbb{B})$
1	20	0.01	3.323	3.284	2.845	0.991	0.997
		0.05	2.593	2.540	2.086	0.955	0.983
		0.1	2.254	2.192	1.724	0.911	0.964
	40	0.01	3.119	3.092	2.705	0.991	0.997
		0.05	2.487	2.443	2.021	0.955	0.983
		0.1	2.183	2.126	1.684	0.912	0.965
	60	0.01	3.056	3.030	2.659	0.991	0.997
		0.05	2.454	2.410	2.000	0.955	0.983
		0.1	2.160	2.103	1.671	0.911	0.965
2	20	0.01	4.050	4.014	3.530	0.991	0.997
		0.05	3.260	3.209	2.722	0.955	0.983
		0.1	2.902	2.839	2.342	0.911	0.965
	40	0.01	3.683	3.660	3.262	0.991	0.997
		0.05	3.045	3.004	2.575	0.954	0.983
		0.1	2.740	2.687	2.238	0.911	0.965
	60	0.01	3.575	3.552	3.185	0.991	0.997
		0.05	2.980	2.942	2.532	0.954	0.983
		0.1	2.691	2.640	2.207	0.911	0.965
5	20	0.01	5.957	5.904	5.266	0.991	0.997
		0.05	4.914	4.847	4.222	0.955	0.983
		0.1	4.448	4.372	3.745	0.911	0.964
	40	0.01	4.920	4.892	4.457	0.991	0.997
		0.05	4.219	4.177	3.711	0.955	0.983
		0.1	3.886	3.834	3.346	0.912	0.965
	60	0.01	4.654	4.634	4.245	0.911	0.997
		0.05	4.034	3.997	3.572	0.955	0.983
		0.1	3.734	3.686	3.235	0.911	0.965

Table 1: Simulation results of $k = 4$

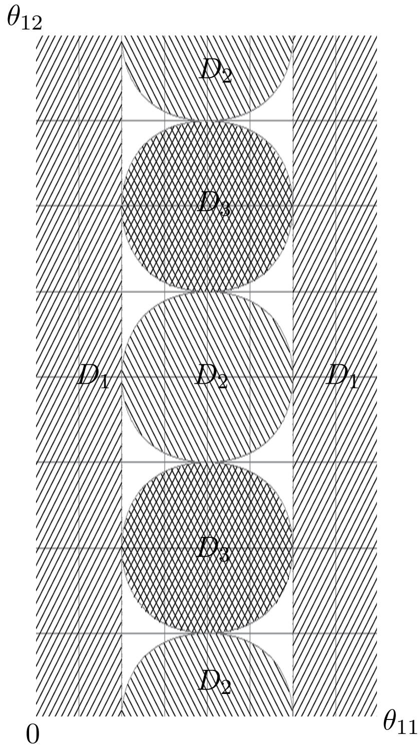

Figure 1. volume$[D_1 \cup D_2 \cup D_3]$ when $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are orthogonal.

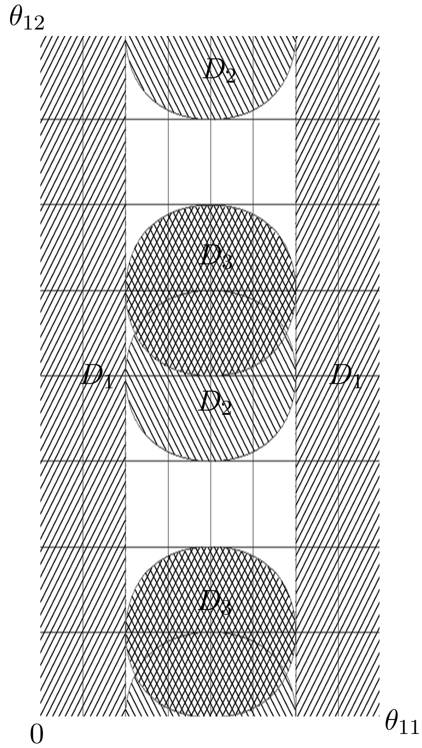

Figure 2. volume$[D_1 \cup D_2 \cup D_3]$ when $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are not orthogonal.

Acknowledgements

The author would like to express his sincere graditude to Professor Takashi Seo for his useful suggestions. The author also would like to thank the referee for his useful comments.

References

[1] Y. Benjamini, and H. Braun, John W. Tukey's contributions to multiple comparisons, *Ann. Statist.* 30 (2002) 1576–1594.

[2] L. D. Brown, A note on the Tukey-Kramer procedure for pairwise comparisons of correlated means, Design of Experiments: Ranking and Selection (Essays in Honorof Robert E. Bechhofer) eds. T. J. Santner and A. C. Tamhane. Marcel Dekker, New York, 1984.

[3] A. J. Hayter, A proof of the conjecture that the Tukey-Kramer multiple comparisons procedure is conservative, *Ann. Statist.* 12 (1984) 61–75.

[4] A. J. Hayter, Pairwise comparisons of generally correlated means, *J. Amer. Statist. Associ.* 84 (1989) 208–213.

[5] Y. Hochberg and A. C. Tamhane, Multiple comparison Procedures, Wiley, New York. 1987.

[6] C. Y. Kramer, Extension of multiple range tests to group means with unequal number of replications, *Biometrics* 12 (1956) 307–310.

[7] C. Y. Kramer, Extension of multiple range tests to group correlated adjusted means, *Biometrics* 13 (1957) 13–18.

[8] T. Nishiyama and T. Seo, The multivariate Tukey-Kramer multiple comparison procedure among four correlated mean vectors, *to appear in Amer. J. Math. Manage. Sci.* (2008).

[9] T. Seo, Simultaneous confidence procedures for multiple comparisons of mean vectors in multivariate normal populations, *Hiroshima Math. J.* 25 (1995) 387–422.

[10] T. Seo, A note on the conservative multivariate Tukey-Kramer multiple comparison procedure, *Amer. J. Math. Manage. Sci.* 16 (1996) 251–266.

[11] T. Seo, S. Mano and Y. Fujikoshi, A generalized Tukey conjecture for multiple comparisons among mean vectors, *J. Amer. Statist. Associ.* 89 (1984) 676–679.

[12] T. Seo and T. Nishiyama, On the conservative simultaneous confidence procedures for multiple comparisons among mean vectors, *to appear in J. Statist. Plann. Infer.* (2007).

[13] M. Siotani, T. Hayakawa and Y. Fujikoshi, Modern Multivariate Analysis : A Graduate Course and Handbook, American Sciences Press, Ohio. 1985.

[14] J. W. Tukey, The problem of multiple comparisons, *Unpublished manuscript, Princeton University* (1953).

Takahiro Nishiyama
Department of Mathematical Information Science, Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: `j1105706@ed.kagu.tus.ac.jp`